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Modeling the backscattering by an elongated deformed body with an elliptical cross section – analytical and closed formed approach based on the Deformed Wave Born Approximation (DWBA)

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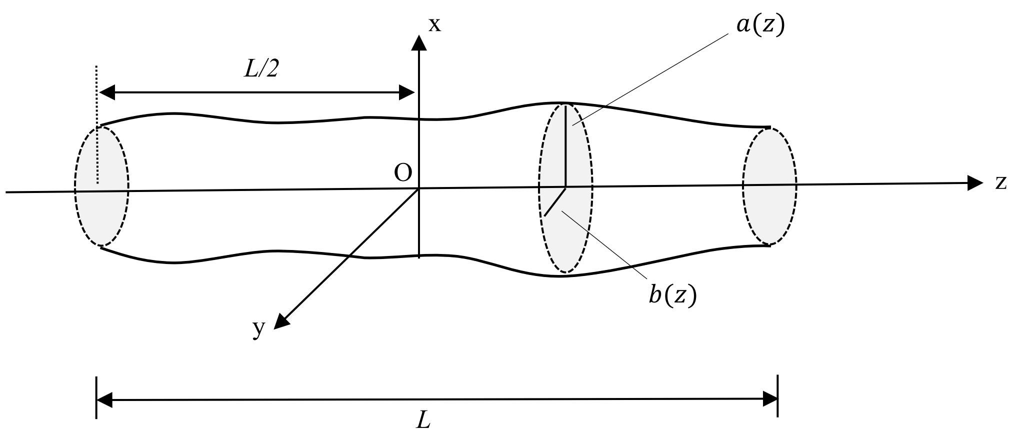

Figure 1. Geometry of the scattering by an elongated object of length *L* with an elliptical cross-section.

The backscattering of a weakly scattering object can be approximated by an integral form of a volume (v_0) based on the Distorted Wave Born Approximation (DWBA) (Chu *et al.* 1993):

$$f_{scat}^{DWBA} = \frac{k_1^2}{4\pi}\int_{\mathrm{v}_0}\left(h^2\gamma_k + \hat{\mathbf{k}}_\mathbf{i}\cdot\hat{\mathbf{k}}_s\gamma_\rho\right)e^{ik_2(\hat{\mathbf{k}}_\mathbf{i}-\hat{\mathbf{k}}_s)\cdot \boldsymbol{r}_\mathbf{0}}\,d\mathrm{v}_0 \qquad (1)$$

where $k_{1,2}$ are the wave numbers in media 1 (surrounding water) and 2 (scatterer), respectively. $\mathbf{r_0}$ is the position inside the scatterer, and $\widehat{\mathbf{k}}_\mathbf{i}$ and $\widehat{\mathbf{k}}_\mathbf{s}$ are unit vectors of the incident and scattered wavenumbers, respectively. $h = k_1 / k_2$ is the sound speed contrast of the animal. By using a density contrast $g = \rho_2 / \rho_1$, we can express γ_k and γ_ρ as

$$\gamma_k = \frac{1-gh^2}{gh^2} \qquad \text{and} \qquad \gamma_\rho = \frac{g-1}{g} \qquad (2)$$

The two unit vectors are

$$k_i = \cos\theta_i \cos\phi_i \hat{x} + \cos\theta_i \sin\phi_i \hat{y} + \sin\theta_i \hat{z} \tag{3}$$

$$k_s = \cos\theta_s \cos\phi_s \hat{x} + \cos\theta_s \sin\phi_s \hat{y} + \sin\theta_s \hat{z} \tag{4}$$

where (θ, ϕ) are spherical angles (polar and azimuth) in the spherical coordinates.

Note that in general the parameters of wave number, k_2, sound speed contrast, h_2, relative differences of compressibility and density, γ_k and γ_k, are all functions of positions inside the integration volume, v_0. For a homogeneous scatter, g and h are constant inside the scatterer, we can express Eq. (1) as

$$f_{scat}^{DWBA} = \frac{k_1^2}{4\pi}\left(h^2\gamma_k + \hat{\mathbf{k}}_\mathbf{i} \cdot \hat{\mathbf{k}}_s \gamma_\rho\right) \int_{\mathrm{v}_0} e^{ik_2(\hat{\mathbf{k}}_\mathbf{i} - \hat{\mathbf{k}}_s)\cdot \boldsymbol{r}_\mathbf{0}}\, d\mathrm{v}_0 = C_s I \tag{5}$$

The constant term:

$$C_s = \frac{k_1^2}{4\pi}(h^2\gamma_k + \hat{\mathbf{k}}_\mathbf{i} \bullet \hat{\mathbf{k}}_\mathbf{s} \gamma_\rho) \tag{6}$$

and the integral term

$$\begin{aligned} I &= \int_{v_0} e^{ik_2(\hat{\mathbf{k}}_\mathbf{i} - \hat{\mathbf{k}}_\mathbf{s})\bullet \mathbf{r}_\mathbf{0}}\, dv_0 \\ &= \int_{-c}^{c} dz_0 \int_0^{r_a(z_0)} r_0 dr_0 \int_0^{2\pi} e^{ik_2(\hat{\mathbf{k}}_\mathbf{i} - \hat{\mathbf{k}}_\mathbf{s})\bullet \mathbf{r}_\mathbf{0}}\, d\phi_0 \end{aligned} \tag{7}$$

where the upper limit of the radial integral is a function of z_0. The phase of the integrand is

$$\begin{aligned} \hat{\mathbf{k}}_\mathbf{i} - \hat{\mathbf{k}}_\mathbf{s} = &(\cos\theta_i \cos\phi_i - \cos\theta_s \cos\phi_s)\hat{x} + \\ &(\cos\theta_i \sin\phi_i - \cos\theta_s \sin\phi_s)\hat{y} + \\ &(\sin\theta_i - \sin\theta_s)\hat{z} \end{aligned} \tag{8}$$

Hence

$$\begin{aligned} \left(\hat{\mathbf{k}}_\mathbf{i} - \hat{\mathbf{k}}_\mathbf{s}\right) \cdot \boldsymbol{r}_\mathbf{0} = &(\cos\theta_i \cos\phi_i - \cos\theta_s \cos\phi_s)x_0 + \\ &(\cos\theta_i \sin\phi_i - \cos\theta_s \sin\phi_s)y_0 + \\ &(\sin\theta_i - \sin\theta_s)z_0 \\ = &\ \Pi_x x_0 + \Pi_y y_0 + \Pi_z z_0 \end{aligned} \tag{9}$$

Note that in the X-Y plane or the polar plane (r_0, ϕ_0), for an ellipse, the integral upper limit in radial direction should be a function of azimuth angle ϕ_0, which makes the integral very difficult to perform analytically. To make the integral more manageable, consider an elliptical cross section of the object in X-Y plane along z-axis,

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1, \tag{10}$$

where $a = a(z_0)$ and $b = b(z_0)$ are the semi-major/semi-minor of the ellipse and are functions of z_0. Define a new normalized variable y_{0n} related to the original y_0 as $y_0 = y_{0n}\frac{b}{a} = y_{0n}e_{ba}$, where e_{ba} is the aspect ratio of the ellipse in the X-Y plane, (10) becomes a circle

$$\frac{x_0^2}{a^2}+\frac{y_{0n}^2}{a^2}=1 \tag{11}$$

Substituting $y_0 = y_{0n}e_{ba}$ into (9) leads to

$$(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\bullet\mathbf{r}_0=\Pi_{\mathrm{x}}x_0+e_{ba}\Pi_{\mathrm{y}}\mathrm{y}_{0\mathrm{n}}+\Pi_{\mathrm{z}}\mathrm{z}_0 \tag{12}$$

Define $r_{0n} = \sqrt{x_0^2 + y_{0n}^2}$ with $x_0 = r_{0n}\cos\phi_0$ and $y_{0n} = \mathrm{r}_{0\mathrm{n}}\sin(\phi_0)$, re-arrange (12) and using the relations of trigonometry, we have

$$\begin{aligned}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{\mathbf{s}})\cdot\mathbf{r}_0 &= \Pi_{\mathrm{x}}x_0+e_{ba}\Pi_{\mathrm{y}}y_{0n}+\Pi_{\mathrm{z}}z_0\\ &= r_{0n}\Pi_{\mathrm{x}}\cos\phi_0+e_{ba}r_{0n}\Pi_{\mathrm{y}}\sin\phi_0+\Pi_{\mathrm{z}}z_0\\ &= r_{0n}\sqrt{\Pi_{\mathrm{x}}^2+e_{ba}^2\Pi_{\mathrm{y}}^2}\left(\frac{\Pi_{\mathrm{x}}\cos\phi_0}{\sqrt{\Pi_{\mathrm{x}}^2+e_{ba}^2\Pi_{\mathrm{y}}^2}}+\frac{e_{ba}\Pi_{\mathrm{y}}\sin\phi_0}{\sqrt{\Pi_{\mathrm{x}}^2+e_{ba}^2\Pi_{\mathrm{y}}^2}}\right)+\Pi_{\mathrm{z}}z_0\\ &= r_{0n}\sqrt{\Pi_{\mathrm{x}}^2+e_{ba}^2\Pi_{\mathrm{y}}^2}(\cos\phi'\cos\phi_0+\sin\phi'\sin\phi_0)+\Pi_{\mathrm{z}}z_0\\ &= r_{0n}\sqrt{\Pi_{\mathrm{x}}^2+e_{ba}^2\Pi_{\mathrm{y}}^2}\cos(\phi_0-\phi')+\Pi_{\mathrm{z}}z_0\end{aligned} \tag{13}$$

where

$$\phi' = \tan^{-1}\frac{e_{ba}\Pi_y}{\Pi_x} \tag{14}$$

With the upper limit of the integral Eq. (7) of $r_a(z_0)$, the integral given by Eq. (7) becomes independent of ϕ_0 and can be expressed as

$$I = \int_{-L/2}^{L/2} dz_0 \int_0^{r_a(z_0)} r_{0n} dr_{0n} \int_0^{2\pi} e^{ik_2(\hat{\mathbf{k}}_\mathbf{i}-\hat{\mathbf{k}}_\mathbf{s})\cdot\mathbf{r}_0}\, d\phi_0$$
$$= e_{ba}\int_{-L/2}^{L/2} dz_0 \int_0^{r_a(z_0)} r_{0n} dr_{0n} \int_0^{2\pi} e^{ik_2 r_{0n}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\cos(\phi_0-\phi')+ik_2\Pi_z z_0} d\phi_0 \tag{15}$$

Note that the integral over azimuth angle is (Abramowitz & Stegun, 1970, p360, 9.1.18)

$$I_\phi = \int_0^{2\pi} e^{ik_2 r_{0n}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\cos(\phi_0-\phi')} d\phi_0 = 2\pi J_0\left(k_2 r_{0n}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right), \tag{16}$$

where $J_0(x)$ is the cylindrical Bessel function of order zero. Substituting Eq. (16) into (15) leads to

$$I = 2\pi e_{ba}\int_{-L/2}^{L/2} e^{i\Pi_z z_0} dz_0 \int_0^{r_a(z_0)} r_{0n} J_0\left(k_2 r_{0n}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) dr_{0n} \tag{17}$$

From Gradshtyen and Ryzhik (1980, p634, 5.52.1), the radial integration can be expressed as

$$I_{r_{0n}} = \int_0^{r_a(z_0)} r_{0n} J_0\left(k_2 r_{0n}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) dr_{0n}$$
$$= \frac{r_a(z_0)}{k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} J_1\left(k_2 a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) \tag{18}$$

Substituting (18) into (17), we obtain

$$I = \frac{2\pi e_{ba}}{k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}\int_{-L/2}^{L/2} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0} dz_0 \tag{19}$$

Therefore, a general 1-D integral representation of the backscattering by an object with an arbitrarily deformed lengthwise axis and an elliptical cross section can be obtained by substituting (6) and (19) to (5):

$$f_{scat}^{DWBA} = \frac{k_1^2}{2}\int_{-L/2}^{L/2} \frac{e_{ba}(h^2\gamma_k-\gamma_\rho) r_a(z_0)}{k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0} dz_0 \tag{20}$$

If $k_2, e_{ba}, \gamma_k,$ and γ_ρ are independent of z, Eq. (20) can be reduced to

$$f_{scat}^{DWBA} = \frac{e_{ba}k_1^2(h^2\gamma_k-\gamma_\rho)}{2k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}\int_{-L/2}^{L/2} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0} dz_0 \tag{21}$$

Equations (21) can be reduced to analytical and closed form solutions for objects with simpler geometries:

Case 1: Scatterng by a homogeneous ellipsoid

For an ellipsoid shown in Fig. 2, we have

$$r_a(z_0) = a\sqrt{1 - \left(\frac{z_0}{c}\right)^2}, \tag{22}$$

where $e_{ca} = \frac{c}{a}$ is the ratio of two semi-axes and $L = 2c$.

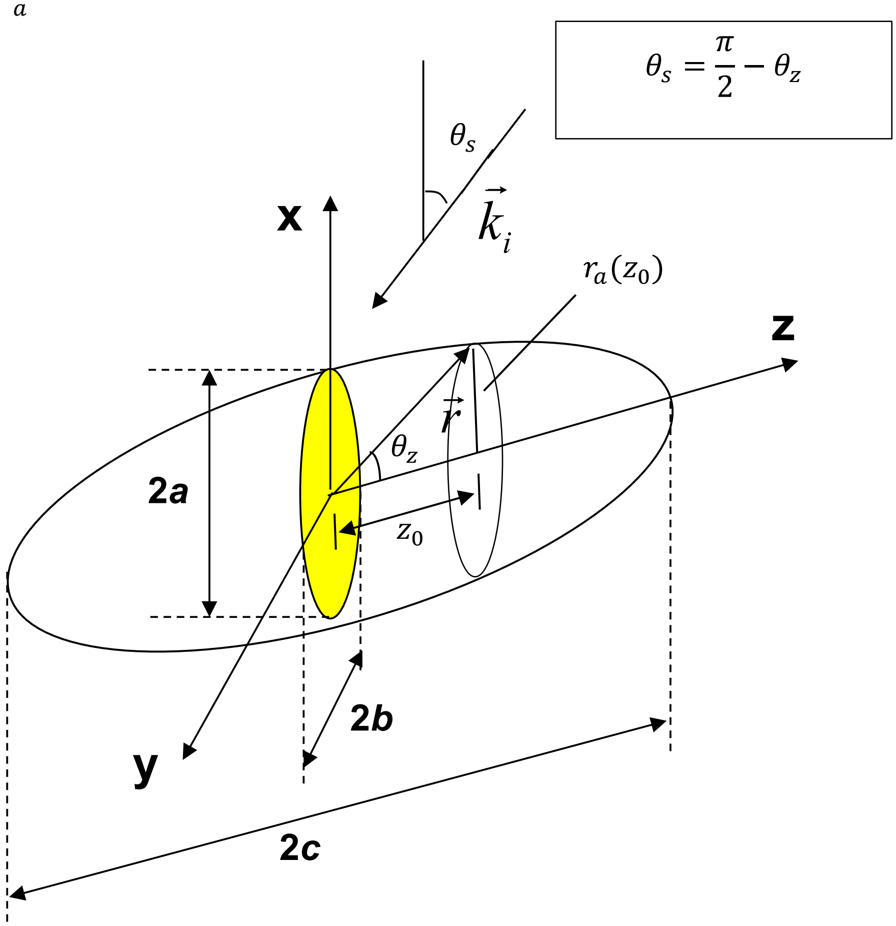

Figure 2. Geometry of an ellipsoid.

Substituting (22) into (21) leads to

$$
\begin{aligned}
I_{z_0} &= \int_{-L/2}^{L/2} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0} dz_0 \\
&= a\int_{-L/2}^{L/2} \sqrt{1-\left(\frac{z_0}{c}\right)^2} J_1\left(k_2 a\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\sqrt{1-\left(\frac{z_0}{c}\right)^2}\right) e^{ik_2\Pi_z z_0} dz_0 \\
&= ac\int_{-1}^{1} \sqrt{1-u^2} J_1\left(k_2 a\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\sqrt{1-u^2}\right) e^{ik_2 c\Pi_z u} du \\
&= 2k_2 a^2 c\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\,\frac{j_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{ca}^2\Pi_z^2}\right)}{k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{ca}^2\Pi_z^2}}
\end{aligned}
\tag{23}
$$

from the 3rd to the 4th line, we used the result from Erdelyi et al. [1954, vol. I p57, (50), $a = 1, \nu = 1, y = k_2 c\Pi_z, b = k_2 a\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}$]. The function $j_1(x)$ is the Spherical Bessel function of order 1 and is related to the Cylindrical Bessel function by (Abramowitz & Stegun, 1970, p437, 10.1.1)

$$
j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \tag{24}
$$

Substituting (23) into (21), we have

$$
f_{scat}^{DWBA} = k_1^2(h^2\gamma_k - \gamma_\rho)abc\,\frac{j_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{ca}^2\Pi_z^2}\right)}{k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{ca}^2\Pi_z^2}} \tag{25}
$$

Equation (26) is valid for a weakly scattering object with arbitrary directions for both the incident and scattered waves.

For backscattering scenario, which is the case for most our applications, $\hat{\mathbf{k}}_\mathbf{i} \bullet \hat{\mathbf{k}}_\mathbf{s} = -1$, $\theta_i = \pi - \theta_s$ and $\phi_i = \pi + \phi_s$, the three sine and cosine items in Eq. (9) are

$$
\begin{aligned}
\Pi_x &= -2\cos\theta_s\cos\phi_s \\
\Pi_y &= -2\cos\theta_s\sin\phi_s \\
\Pi_z &= -2\sin\theta_s
\end{aligned}
\tag{26}
$$

Note that $\theta_s = 0$ corresponds to broadside incidence, with $\phi_s = 0$, for incident wave along x-axis while $\phi_s = \frac{\pi}{2}$ for incident wave along y-axis.

Now we investigate a few special cases:

(a) $\phi_i = 0$ (incidence in XZ plane), Eq. (26) reduces to

$$f_{scat}^{DWBA} = k_1^2(h^2\gamma_k - \gamma_\rho)abc\frac{j_1(2k_2a\Theta_{\theta,0})}{2k_2a\Theta_{\theta,0}} \tag{27}$$

with

$$\Theta_{\theta,0} = \sqrt{\cos^2\theta_s + e_{ca}^2\sin^2\theta_s} \tag{28}$$

(b) $\phi_i = \frac{\pi}{2}$ (incidence in Y-Z plane),

$$f_{scat}^{DWBA} = k_1^2(h^2\gamma_k - \gamma_\rho)abc\frac{j_1(2k_2b\Theta_{\theta,\frac{\pi}{2}})}{2k_2b\Theta_{\theta,\frac{\pi}{2}}} \tag{29}$$

$$\Theta_{\theta,\frac{\pi}{2}} = \sqrt{\cos^2\theta_s + e_{cb}^2\sin^2\theta_s} \tag{30}$$

(c) $\theta_s = \frac{\pi}{2}$ (end-on incidence along z-axis)

$$f_{scat}^{DWBA} = k_1^2(h^2\gamma_k - \gamma_\rho)abc\frac{j_1(2k_2c\Theta_{\frac{\pi}{2},\phi})}{2k_2c\Theta_{\frac{\pi}{2},\phi}} \tag{31}$$

$$\Theta_{\frac{\pi}{2},\phi} = 1 \tag{32}$$

(d) $\theta_s = 0$ (broadside incidence, along x-axis)

$$f_{scat}^{DWBA} = k_1^2(h^2\gamma_k - \gamma_\rho)abc\frac{j_1(2k_2a\Theta_{0,\phi})}{2k_2a\Theta_{0,\phi}} \tag{33}$$

$$\Theta_{0,\phi} = \sqrt{\cos^2\phi_s + e_{ba}^2\sin^2\phi_s} \tag{34}$$

Case 2: <u>Scatterng by a homogeneous straight elongated object with an elliptical cross-section and flat ends</u>

In this case, $r_a(z_0) = a$ in Eq. (21), and the integration is straightforward and can be easily performed to produce:

$$f_{scat}^{DWBA} = \frac{k_1^2(h^2\gamma_k-\gamma_\rho)bL}{4k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}J_1\left(k_2a\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right)\frac{\sin\left(\frac{k_2L}{2}\Pi_z\right)}{\frac{k_2L}{2}\Pi_z} \tag{35}$$

<u>Case 3: Case 2 but with half ellipsoids on both ends</u>

In this case, we set $L = L_0 + c_1 + c_2$, where c_1 and c_2 are semi-axes along z-axis for front and back ends, respectively as shown in Fig. 3

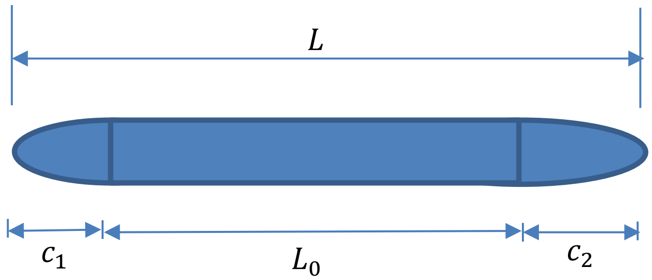

Figure 3. Geometry of a homogeneous straight elongated object with an elliptical cross-section and half ellipsoids on both ends.

$$
\begin{aligned}
f_{scat}^{DWBA} &= \frac{e_{ba}k_1^2(h^2\gamma_k-\gamma_\rho)}{2k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0}\, dz_0 \\
&= \frac{e_{ba}k_1^2(h^2\gamma_k-\gamma_\rho)}{2k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} \left(I_{c_1}+I_{L_0}+I_{c_2}\right)
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
I_{L_0} &= \int_{-\frac{L_0}{2}}^{\frac{L_0}{2}} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0}\, dz_0 \\
&= \frac{L_0}{2} J_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) \frac{\sin\left(\frac{k_2L_0}{2}\Pi_z\right)}{\frac{k_2L_0}{2}\Pi_z}
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
I_{c_1} &= \int_{-L/2}^{-L_0/2} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0}\, dz_0 \\
&= e^{-\frac{ik_2\Pi_z L_0}{2}} \int_{-c_1}^{0} r_a(z) J_1\left(k_2 r_a(z)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z}\, dz \\
&= -ac_1 e^{-\frac{ik_2\Pi_z L_0}{2}} \int_0^1 \sqrt{1-u^2}\, J_1\left(k_2 a\sqrt{1-u^2}\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right) e^{-ik_2c_1\Pi_z u}\, du \\
&= -k_2 a^2 c_1 e^{-\frac{ik_2\Pi_z L_0}{2}} \sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\, \frac{j_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_1a}^2\Pi_z^2}\right)}{k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_1a}^2\Pi_z^2}}
\end{aligned} \tag{38}
$$

$$I_{c_2} = \int_{L_0/2}^{L/2} r_a(z_0) J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z_0} dz_0$$
$$= e^{\frac{ik_2\Pi_z L_0}{2}} \int_0^{c_2} r_a(z) J_1\left(k_2 r_a(z)\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right) e^{ik_2\Pi_z z} dz$$
$$= ac_2 e^{\frac{ik_2\Pi_z L_0}{2}} \int_0^1 \sqrt{1-u^2} J_1\left(k_2 a\sqrt{1-u^2}\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right) e^{ik_2 c_2\Pi_z u} du \quad (38)$$
$$= k_2 a^2 c_2 e^{\frac{ik_2\Pi_z L_0}{2}} \sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\, \frac{j_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_2a}^2\Pi_z^2}\right)}{k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_2a}^2\Pi_z^2}}$$

For $c_1 = c_2 = c$, Eq. (36) has a simpler form:

$$f_{scat}^{DWBA} = \frac{e_{ba}k_1^2(h^2\gamma_k - \gamma_\rho)}{2k_2\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}\left[\frac{L_0}{2} J_1\left(k_2 a\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\right)\frac{\sin\left(\frac{k_2 L_0}{2}\Pi_z\right)}{\frac{k_2 L_0}{2}\Pi_z} + \right.$$
$$\left. + 2k_2 a^2 c\sqrt{\Pi_x^2 + e_{ba}^2\Pi_y^2}\,\frac{j_1\left(k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_2a}^2\Pi_z^2}\right)}{k_2 a\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2+e_{c_2a}^2\Pi_z^2}}\cos\left(\frac{k_2\Pi_z L_0}{2}\right)\right] \quad (39)$$

Case 4: Scattering by a homogeneous uniformly bent elongated object with an elliptical cross-section

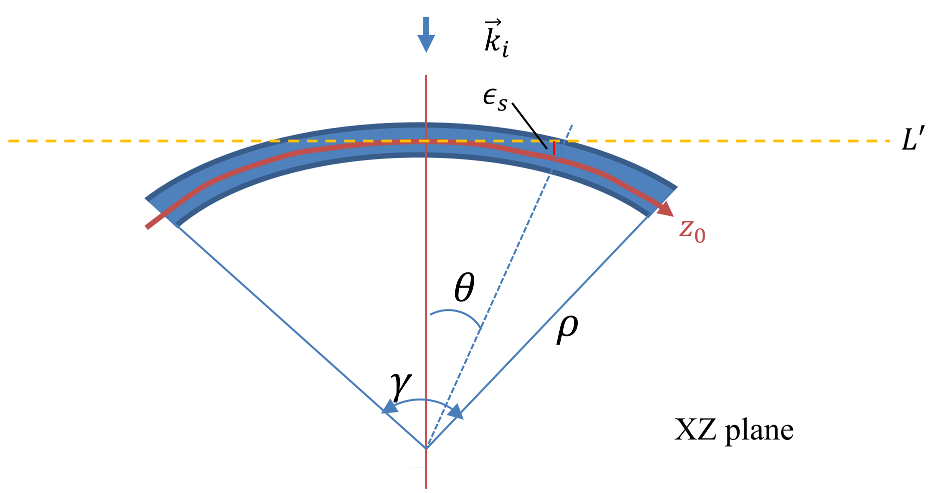

Figure 4. Geometry of scattering by a homogeneous uniformly bent elongated object with an elliptical cross-section

Without loss of generality, we assume the incident wave is in the plane of bending. In this case, the angle θ is a function of z_0, $\theta(z_0)$, with $\theta\left(\pm\frac{L}{2}\right) = \pm\frac{\gamma}{2}$. The two variables Π_x and Π_y are functions of z_0. In addition, a phase correction term, $e^{i\Delta\varphi}$, with (Stanton, 1989)

$$\Delta\varphi = 2k_1\epsilon_s = 2k_1\rho(1-\cos\theta) = 2k_1\rho\left(1-\cos(\frac{\gamma}{L}z_0)\right)$$

needs to be added to Eq. (21) to account for the deflection, ϵ_s ,of the object axis away from the horizontal line, L'. In addition, since Π_x and Π_y are functions of z_0, the square root term in the denominator of Eq. (21) will be part of the integrand

$$\begin{aligned} I_{z_0} &= \int_{-L/2}^{L/2} r_a(z_0)\frac{J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right)}{\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} e^{ik_2\Pi_z z_0} e^{i2k_1\rho\epsilon_s(z_0)}dz_0 \\ &= a\int_{-L/2}^{L/2}\frac{J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right)}{\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}} e^{ik_2\Pi_z z_0} e^{i2k_1\rho\left(1-\cos(\frac{\gamma}{L}z_0)\right)}dz_0 \end{aligned} \tag{40}$$

For $k_1\rho \gg 1$, we can evaluate above integral by the method of stationary phase (Wong, 2001). The stationary phase point can be obtained by taking the 1st derivative of the phase term and setting it to zero

$$\frac{\partial\Phi}{\partial z_0} = k_2\Pi_z + 2k_1\rho\frac{\gamma}{L}\sin\left(\frac{\gamma}{L}z_0\right) = 0$$

Which leads to

$$z_s = -\frac{L}{\gamma}\sin^{-1}\left(\frac{k_2\Pi_z L}{2k_1\rho\gamma}\right) \tag{41}$$

and

$$\left.\frac{\partial^2\Phi}{\partial z_0^2}\right|_{z_0=z_s} = 2k_1\rho\left(\frac{\gamma}{L}\right)^2\cos\left(\frac{\gamma}{L}z_s\right) > 0 \tag{42}$$

$$\begin{aligned} I_{z_0} &= k_2 a\frac{J_1\left(\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right)}{\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}\sqrt{\frac{2\pi}{\left.\frac{\partial^2\Phi}{\partial z_0^2}\right|_{z_0=z_s}}}\, e^{i\left[\Phi(z_s)+\frac{\pi}{4}\right]} \\ &= \frac{k_2 a\rho}{\gamma}\frac{J_1\left(\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}\right)}{\sqrt{\Pi_x^2+e_{ba}^2\Pi_y^2}}\sqrt{\frac{\pi}{k_1\rho\cos\left(\frac{\gamma}{L}z_s\right)}}\, e^{i\left(k_2\Pi_z z_s+\left[2k_1\rho\, 1-\cos\left(\frac{\gamma}{L}z_s\right)\right]+\frac{\pi}{4}\right)} \end{aligned} \tag{43}$$

$$f_{scat}^{DWBA} = \frac{a\rho e_{ba} k_1^2 (h^2\gamma_k - \gamma_\rho)}{2\gamma\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}} J_1\left(\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}\right) \sqrt{\frac{\pi}{k_1 \rho \cos\left(\frac{\gamma}{L} z_s\right)}}\, e^{i\left(k_2 \Pi_z z_s + \left[2k_1\rho\, 1 - \cos\left(\frac{\gamma}{L} z_s\right)\right] + \frac{\pi}{4}\right)}$$

(44)

<u>Case 5. Case4 but with arbitrarily bent and non-flat ends on both sides, a more general case</u>

Equation (20) is actually a general form of the scattering by an arbitrarily bent elongated object with an elliptical cross-section and non-flat ends on both sides. Assuming a target of homogenous material properties, Equation (20) can be modified to include the phase deviation resulting from lengthwise axis away from the mean horizontal position described in Case 4:

$$f_{scat}^{DWBA} = \frac{k_1^2 (h^2\gamma_k - \gamma_\rho)}{2k_2} \int_{-L/2}^{L/2} r_a(z_0) e_{ba}(z_0) \frac{J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}\right)}{\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}} e^{ik_2 \Pi_z z_0} e^{i2k_1 \epsilon_s(z_0)} dz_0$$

$$= \frac{k_1^2 (h^2\gamma_k - \gamma_\rho)}{2} \int_{-L/2}^{L/2} r_a(z_0) r_b(z_0) \frac{J_1\left(k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}\right)}{k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2}} e^{ik_2 \Pi_z z_0} e^{i2k_1 \epsilon_s(z_0)} dz_0$$

$$= \frac{k_1^2 (h^2\gamma_k - \gamma_\rho)}{2} \int_{-L/2}^{L/2} r_a(z_0) r_b(z_0) \frac{J_1(\Xi(z_0))}{\Xi(z_0)} e^{ik_2 \Pi_z z_0} e^{i2k_1 \epsilon_s(z_0)} dz_0$$

(45)

where

$$\epsilon_s(z_0) = x(z_0) \quad (46)$$

$$\Xi(z_0) = k_2 r_a(z_0)\sqrt{\Pi_x^2 + e_{ba}^2 \Pi_y^2} \quad (47)$$

With $[\theta_i = \pi - \theta_s \text{ in Eq.} (9)]$

$$\begin{aligned} \Pi_x &= -2\cos\theta_s \cos\phi_s \\ \Pi_y &= -2\cos\theta_s \sin\phi_s \\ \Pi_z &= -2\sin\theta_s \end{aligned} \quad (48)$$

Case 6: Scattering by a homogeneous ellipsoidal fluid ellipsoidal shell (Concentered)

The backscattering by a weakly scattering and fluid ellipsoidal shell (Figure obtained from Eq. (1)

$$
\begin{aligned}
f_{scat}^{DWBA} &= \frac{k_w^2}{4\pi}\int_{\mathrm{V}_0=V_{shell}+V_{in}}\left(h^2\gamma_k + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_{s}\gamma_\rho\right)e^{ik_s(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{s})\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 \\
&= \frac{k_w^2}{4\pi}\int_{V_{shell}}\left(h_{sh}^2\gamma_{k_{sh}} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_{s}\gamma_{\rho_{sh}}\right)e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{s})\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 \\
&\quad + \frac{k_w^2}{4\pi}\int_{V_{in}}\left(h_f^2\gamma_{k_f} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_{s}\gamma_{\rho_f}\right)e^{ik_f(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{s})\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0
\end{aligned}
\quad , \qquad (49)
$$

where $k_w = \frac{2\pi f}{c_w},\ k_{sh} = \frac{2\pi f}{c_{shell}},\ k_f = 2\pi f/c_f$.

For a uniform shell and fluid inside shell, above equation can be generalized to

$$
\begin{aligned}
f_{scat}^{DWBA} = &\frac{k_w^2}{4\pi}\left(h_{sh}^2\gamma_{k_{sh}} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_{s}\gamma_{\rho_{sh}}\right)\int_{V_{shell}} e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{s})\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 \\
&+ \frac{k_w^2}{4\pi}\left(h_f^2\gamma_{k_f} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_{s}\gamma_{\rho_f}\right)\int_{V_{in}} e^{ik_f(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_{s})\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0
\end{aligned}
\qquad (50)
$$

Assuming the sound speed and density contrasts of the water/shell and shell/inside interfaces are

$$
\begin{aligned}
h_{sh} = \frac{c_{shell}}{c_w},\ g_{sh} = \frac{\rho_{shell}}{\rho_w} \\
h_f = \frac{c_f}{c_w},\ g_f = \frac{\rho_f}{\rho_w}
\end{aligned}
\quad . \qquad (51)
$$

Therefore, from Eq. (2), we have

$$
\gamma_{k_{sh}} = \frac{1-g_{sh}h_{sh}^2}{g_{sh}h_{sh}^2}, \qquad \gamma_{k_f} = \frac{1-g_f h_f^2}{g_f h_f^2}, \qquad \gamma_{\rho_{sh}} = \frac{1-g_{sh}}{g_{sh}}, \qquad \gamma_{\rho_f} = \frac{1-g_f}{g_f}. \qquad (52)
$$

The backscattering from this fluid shell can be derived easily from Eq. (25):

$$
\begin{aligned}
f_{scat}^{DWBA} = &k_w^2\left(h_{sh}^2\gamma_{k_{sh}} - \gamma_{\rho_{sh}}\right)\Bigg\{a_{sh}b_{sh}c_{sh}\frac{j_1\left(k_{sh}a_{sh}\sqrt{\Pi_\mathrm{x}^2+e_{b_{sh}a_{sh}}^2\Pi_\mathrm{y}^2+e_{c_{sh}a_{sh}}^2\Pi_\mathrm{z}^2}\right)}{k_{sh}a_{sh}\sqrt{\Pi_\mathrm{x}^2+e_{b_{sh}a_{sh}}^2\Pi_\mathrm{y}^2+e_{c_{sh}a_{sh}}^2\Pi_\mathrm{z}^2}} \\
&- a_f b_f c_f\frac{j_1\left(k_{sh}a_f\sqrt{\Pi_\mathrm{x}^2+e_{b_f a_f}^2\Pi_\mathrm{y}^2+e_{c_f a_f}^2\Pi_\mathrm{z}^2}\right)}{k_{sh}a_f\sqrt{\Pi_\mathrm{x}^2+e_{b_f a_f}^2\Pi_\mathrm{y}^2+e_{c_f a_f}^2\Pi_\mathrm{z}^2}}\Bigg\} \\
&+ k_w^2\left(h_f^2\gamma_{k_f} - \gamma_{\rho_f}\right)a_f b_f c_f\frac{j_1\left(k_f a_f\sqrt{\Pi_\mathrm{x}^2+e_{b_f a_f}^2\Pi_\mathrm{y}^2+e_{c_f a_f}^2\Pi_\mathrm{z}^2}\right)}{k_f a_f\sqrt{\Pi_\mathrm{x}^2+e_{b_f a_f}^2\Pi_\mathrm{y}^2+e_{c_f a_f}^2\Pi_\mathrm{z}^2}}
\end{aligned}
\quad , \qquad (53)
$$

where $a_f \le a_{sh}$, $b_f = a_f e_{b_f a_f} \le b_{sh}$, and $c_f = a_f e_{c_f a_f} \le c_{sh}$.

Note that in general the outer and inner ellipsoids can have different aspect ratios, i.e. in general $e_{b_{sh}a_{sh}} \neq e_{b_f a_f}$ and $e_{c_{sh}a_{sh}} \neq e_{c_f a_f}$ although they can be the same. The 1st term in the curly bracket is the contribution from the fluid shell while the 2nd term is the contribution from the inside fluid.

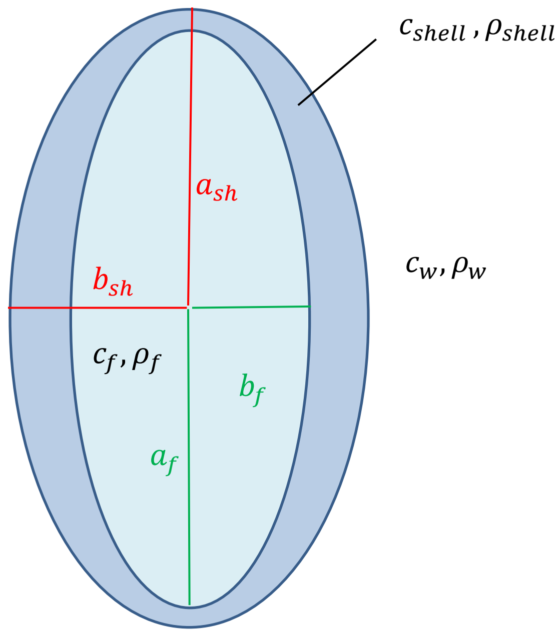

Figure 5. 2D geometry of a fluid ellipsoidal shell with both shell and inside fluid are weakly scattering materials. Note that in general the outer and inner ellipsoids can have different aspect ratios, i.e. in general $e_{b_{sh}a_{sh}} \neq e_{b_f a_f}$ and $e_{c_{sh}a_{sh}} \neq e_{c_f a_f}$ although they can be the same.

Case 7: <u>Scattering by a homogeneous ellipsoidal fluid ellipsoidal shell (Non-concentered)</u>

For a non-concentered case (centers of the two ellipsoids in Fig. 5 are at different coordinates), in Eq. (9), the vector $\boldsymbol{r_0}$ for the inner ellipsoid can be replaced by a new vector and be expressed as

$$\boldsymbol{r_0} = \boldsymbol{r_s} + \boldsymbol{r_0}', \tag{54}$$

where a constant offset or shift vector $\boldsymbol{r_s}$ is defined as

$$\boldsymbol{r_s} = x_s\hat{x} + y_s\hat{y} + z_s\hat{z} \tag{55}$$

Equation (50) for a non-concenter scenario will become

$$f_{scat}^{DWBA} = \frac{k_w^2}{4\pi}\left(h_{sh}^2\gamma_{k_{sh}} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_s\gamma_{\rho_{sh}}\right)\left[\int_{V_{outer}} e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 - \int_{V_{inner}} e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0\right] + \frac{k_w^2}{4\pi}\left(h_f^2\gamma_{k_f} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_s\gamma_{\rho_f}\right)\int_{V_{inner}} e^{ik_f(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 \tag{56}$$

Substituting $\boldsymbol{r_0}$ into the 2nd and 3rd terms in Eq. (56) and using Eq. (55), we obtain

$$f_{scat}^{DWBA} = \frac{k_w^2}{4\pi}\left(h_{sh}^2\gamma_{k_{sh}} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_s\gamma_{\rho_{sh}}\right)\left[\int_{V_{outer}} e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}}\,d\mathrm{v}_0 - e^{i\phi_{sh}}\int_{V_{inner}} e^{ik_{sh}(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}'}\,d\mathrm{v}_0\right] + \frac{k_w^2}{4\pi}e^{i\phi_f}\left(h_f^2\gamma_{k_f} + \hat{\mathbf{k}}_{\mathbf{i}}\cdot\hat{\mathbf{k}}_s\gamma_{\rho_f}\right)\int_{V_{inner}} e^{ik_f(\hat{\mathbf{k}}_{\mathbf{i}}-\hat{\mathbf{k}}_s)\cdot\boldsymbol{r_0}'}\,d\mathrm{v}_0 \tag{57}$$

where the two phase terms are

$$\begin{aligned}\phi_{sh} &= k_{sh}\left(\Pi_x x_s + \Pi_y y_s + \Pi_z z_s\right)\\ \phi_f &= k_f\left(\Pi_x x_s + \Pi_y y_s + \Pi_z z_s\right)\end{aligned} \tag{58}$$

It is straightforward to obtain the final expression for the backscattering amplitude:

$$\begin{aligned}f_{scat}^{DWBA} = {}& k_w^2\left(h_{sh}^2\gamma_{k_{sh}} - \gamma_{\rho_{sh}}\right)\Bigg\{a_{sh}b_{sh}c_{sh}\frac{j_1\left(k_{sh}a_{sh}\sqrt{\Pi_x^2+e_{b_{sh}a_{sh}}^2\Pi_y^2+e_{c_{sh}a_{sh}}^2\Pi_z^2}\right)}{k_{sh}a_{sh}\sqrt{\Pi_x^2+e_{b_{sh}a_{sh}}^2\Pi_y^2+e_{c_{sh}a_{sh}}^2\Pi_z^2}}\\ &- a_f b_f c_f\frac{j_1\left(k_{sh}a_f\sqrt{\Pi_x^2+e_{b_f a_f}^2\Pi_y^2+e_{c_f a_f}^2\Pi_z^2}\right)}{k_{sh}a_f\sqrt{\Pi_x^2+e_{b_f a_f}^2\Pi_y^2+e_{c_f a_f}^2\Pi_z^2}}e^{ik_{sh}\left(\Pi_x x_s+\Pi_y y_s+\Pi_z z_s\right)}\Bigg\}\\ &+ k_w^2\left(h_f^2\gamma_{k_f} - \gamma_{\rho_f}\right)a_f b_f c_f\frac{j_1\left(k_f a_f\sqrt{\Pi_x^2+e_{b_f a_f}^2\Pi_y^2+e_{c_f a_f}^2\Pi_z^2}\right)}{k_f a_f\sqrt{\Pi_x^2+e_{b_f a_f}^2\Pi_y^2+e_{c_f a_f}^2\Pi_z^2}}e^{ik_f\left(\Pi_x x_s+\Pi_y y_s+\Pi_z z_s\right)}\end{aligned} \tag{59}$$

References:

Gradshteyn, I. S., Ryzhik, I. M., Geronimus, Yu. V., and Tseytlin, M. Yu. (1980). *Table of Integrals, Series, and* Products. Jeffrey, Alan (ed.).Translated by Scripta Technica, Inc. (4th corrected and enlarged ed.). Academic Press, Inc.

A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G.Tricomi (1954). *Tables of integral transforms. based, in part, on notes* left by Harry Bateman, and compiled by the staff of the bateman manuscript project, McGraw-Hill.

Stanton, T. K. (1989). “Sound scattering by cylinders of finite length. III. Deformed cylinders,” J. Acoust. Soc. Am., 86: 691-705. doi: 10.1121/1.398193

Wong, R. (2001), *Asymptotic Approximations of Integrals*, Classics in Applied Mathematics, Vol. 34. Corrected reprint of the 1989 original. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. xviii+543 pages, ISBN 0-89871-497-4.